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2001 J. Phys. A: Math. Gen. 34 L647

(http://iopscience.iop.org/0305-4470/34/46/102)

J. Phys. A: Math. Gen. **34** (2001) L647–L655 PII: S0305-4470(01)28515-1

LETTER TO THE EDITOR

Exponentially small asymptotics of solutions to the defocusing nonlinear Schrödinger equation: II

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Received 4 September 2001
Published 9 November 2001
Online at stacks.iop.org/JPhysA/34/L647

Abstract
The Riemann–Hilbert problem approach is used to derive a special set of leading-order, exponentially small asymptotics as $t \to \pm\infty$ such that $x/t \to 0^{\pm}$ of solutions to the Cauchy problem for the defocusing nonlinear Schrödinger equation, $\mathrm{i}\partial_t u + \partial_x^2 u - 2(|u|^2 - 1)u = 0$, with finite-density initial data $u(x,0) =_{x\to\pm\infty} \exp\left(\frac{\mathrm{i}(1\mp1)\theta}{2}\right)(1+\mathrm{o}(1))$, $\theta \in [0, 2\pi)$.

PACS numbers: 02.30.Rz, 02.30.Zz

Mathematics Subject Classification: 35Q15, 37K40, 35Q55, 37K15, 30E20, 30E25

The spacetime evolution of the slowly varying amplitude of the complex field envelope in the theory of dark solitons in optical fibres is described by solutions ($u = u(x,t)$) of the Cauchy problem for the defocusing nonlinear Schrödinger equation ($D_f NLSE$) [1],

$$\begin{aligned} &\mathrm{i}\partial_t u + \partial_x^2 u - 2(|u|^2 - 1)u = 0 \qquad (x,t) \in \mathbb{R}\times\mathbb{R} \\ &u(x,0) := u_o(x) \underset{x\to\pm\infty}{=} \exp\left(\frac{\mathrm{i}(1\mp1)\theta}{2}\right)(1+\mathrm{o}(1)) \end{aligned} \tag{1}$$

where $u_o(x) \in \boldsymbol{C}^{\infty}(\mathbb{R})$, $\theta \in [0, 2\pi)$ (see lemma 1) and the o(1) term is to be understood in the sense that, $\forall\,(k,l) \in \mathbb{Z}_{\geqslant 0}\times\mathbb{Z}_{\geqslant 0}$, $|x|^k(\frac{\mathrm{d}}{\mathrm{d}x})^l\left(u_o(x) - \exp\left(\frac{\mathrm{i}(1\mp1)\theta}{2}\right)\right) =_{x\to\pm\infty} 0$.

In this Letter, and in the *solitonless sector*, via the matrix Riemann–Hilbert problem (RHP) approach [2], a special set of exponentially small asymptotics as $t \to \pm\infty$ such that $z_o := x/t \to 0^{\pm}$ of solutions to the Cauchy problem for the $D_f NLSE$ are presented. In the framework of the inverse scattering method (ISM) [3, 4], the $D_f NLSE$ is a completely integrable nonlinear evolution equation (NLEE) [5]. The direct and inverse spectral analyses for several integrable NLEEs from the ZS–AKNS class with non-vanishing asymptotic values

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of the initial data(e) have been considered in [6–16]. Previous works devoted to the asymptotic analysis of the Cauchy problem for the $D_f NLSE$ are due to Its *et al* [17, 18].

The notation used throughout this Letter is summarized:

(1) for a scalar ϖ and a 2×2 matrix Υ, $\varpi^{\mathrm{ad}(\sigma_3)}\Upsilon := \varpi^{\sigma_3}\Upsilon\varpi^{-\sigma_3}$;

(2) for $1 \leqslant p < \infty$ and $\mathcal{D}^{\natural}$ some (point) set, $\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}^{\natural}) := \{f: \mathcal{D}^{\natural} \to \mathrm{M}_2(\mathbb{C});\ \|f(\cdot)\|_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}^{\natural})} := \left(\int_{\mathcal{D}^{\natural}} |f(z)|^p\,|\mathrm{d}z|\right)^{1/p} < \infty\}$ and, for $p = \infty$, $\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}^{\natural}) := \{g: \mathcal{D}^{\natural} \to \mathrm{M}_2(\mathbb{C});\ \|g(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}^{\natural})} := \max_{1\leqslant i,j\leqslant 2}\sup_{z\in\mathcal{D}^{\natural}} |g_{ij}(z)| < \infty\}$, where, for a matrix $\mathcal{A}_{ij}(\cdot)$, $i, j \in \{1, 2\}$, $|\mathcal{A}(\cdot)|$ denotes the Hilbert–Schmidt norm, $|\mathcal{A}(\cdot)| := \left(\sum_{i,j=1}^{2} \overline{\mathcal{A}_{ij}(\cdot)}\,\mathcal{A}_{ij}(\cdot)\right)^{1/2}$, with $\overline{(\cdot)}$ denoting complex conjugation of $(\cdot)$;

(3) $\mathcal{S}_{\mathbb{C}}(D) := \boldsymbol{C}^\infty(D) \cap \{h: D \to \mathbb{C};\ \|h(\cdot)\|_{k,l} := \sup_{x\in\mathbb{R}} |x^k (\frac{\mathrm{d}}{\mathrm{d}x})^l h(x)| < \infty\,\forall\,(k, l) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}\}$, for D an unbounded domain of $\mathbb{R}$.

As the following proposition shows, the $D_f NLSE$ is the zero-curvature condition for the $\mathrm{M}_2(\mathbb{C})$-valued function $\Psi(x, t; \zeta)$.

Proposition 1 [5, 6, 19]. *The necessary and sufficient condition for the compatibility of the following system of linear PDEs (Lax pair), for arbitrary* $\zeta \in \mathbb{C}$,

$$\partial_x \Psi(x, t; \zeta) = \mathcal{U}(x, t; \zeta)\Psi(x, t; \zeta) \qquad \partial_t \Psi(x, t; \zeta) = \mathcal{V}(x, t; \zeta)\Psi(x, t; \zeta) \tag{2}$$

where

$$\mathcal{U}(x, t; \zeta) = -\mathrm{i}\lambda(\zeta)\sigma_3 + \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix}$$

$$\mathcal{V}(x, t; \zeta) = -2\mathrm{i}(\lambda(\zeta))^2\sigma_3 + 2\lambda(\zeta)\begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix} - \mathrm{i}\begin{pmatrix} u\overline{u} - 1 & \partial_x u \\ \partial_x \overline{u} & u\overline{u} - 1 \end{pmatrix}\sigma_3$$

$\lambda(\zeta) := \frac{1}{2}(\zeta + \frac{1}{\zeta})$ *and* $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *with* $\mathrm{tr}(\mathcal{U}(x, t; \zeta)) = \mathrm{tr}(\mathcal{V}(x, t; \zeta)) = 0$, *is that u satisfies the $D_f NLSE$.*

Via the ISM analysis of system (2) in the solitonless sector and a dependent variable transformation (not explicitly written here) $\Psi(x, t; \zeta) \to m(x, t; \zeta)$, one derives the following (normalized at ∞) RHP for the $\mathrm{M}_2(\mathbb{C})$-valued function $m(x, t; \zeta)$.

Lemma 1 [20]. *Let $u(x, t)$ be the solution of the Cauchy problem for the $D_f NLSE$ with initial data* $u(x, 0) := u_o(x) =_{x\to\pm\infty} u_o(\pm\infty)(1 + \mathrm{o}(1))$, *where* $u_o(\pm\infty) := \exp\left(\frac{\mathrm{i}(1\mp 1)\theta}{2}\right)$, $\theta = -\int_{-\infty}^{+\infty} \frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi} \in [0, 2\pi)$, *with $r(\zeta)$ the reflection coefficient associated with the direct scattering problem for the non-self-adjoint Dirac operator (cf system* (2)) $\mathcal{O}^{\mathcal{D}} := \mathrm{i}\sigma_3\partial_x - \begin{pmatrix} \frac{1}{2}(\zeta + \frac{1}{\zeta}) & \mathrm{i}u_o(x) \\ \overline{\mathrm{i}u_o(x)} & \frac{1}{2}(\zeta + \frac{1}{\zeta}) \end{pmatrix}$, $u_o(x) \in \boldsymbol{C}^\infty(\mathbb{R})$ *and* $u_o(x) - u_o(\pm\infty) \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. *Define* $\sigma_c := \{\zeta;\ \mathrm{Im}\,(\zeta) = 0\}$, *oriented from $-\infty$ to $+\infty$. Then* $m(x, t; \zeta): \mathbb{C}\setminus\sigma_c \to \mathrm{M}_2(\mathbb{C})$ *solves the following RHP:*

(i) $m(x, t; \zeta)$ *is piecewise holomorphic* $\forall\,\zeta \in \mathbb{C}\setminus\sigma_c$;

(ii) $m_\pm(x, t; \zeta) := \lim_{\varepsilon\downarrow 0} m(x, t; \zeta \pm \mathrm{i}\varepsilon)$ *satisfy the jump condition*

$$m_+(x, t; \zeta) = m_-(x, t; \zeta)\mathcal{G}(x, t; \zeta) \qquad \zeta \in \mathbb{R}$$

where $\mathcal{G}(x, t; \zeta) := \exp(-\mathrm{i}k(\zeta)(x + 2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\begin{pmatrix} 1 - r(\zeta)\overline{r(\overline{\zeta})} & -\overline{r(\overline{\zeta})} \\ r(\zeta) & 1 \end{pmatrix}$ *and*

$r(\zeta)$ *satisfies* $r(\zeta) =_{\zeta\to 0} \mathcal{O}(\zeta)$, $r(\zeta) =_{\zeta\to\infty} \mathcal{O}(\zeta^{-1})$, $r(\frac{1}{\zeta}) = -\overline{r(\overline{\zeta})}$ *and* $r(\zeta) \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}) \cap \{h(z);\ \|h(\cdot)\|_{\mathcal{L}^\infty(\mathbb{R})} := \sup_{z\in\mathbb{R}} |h(z)| < 1\}$;

(iii) $\det(m(x,t;\zeta))|_{\zeta=\pm1}=0$;

(iv) $m(x,t;\zeta) =_{\zeta\to0} \frac{1}{\zeta}\sigma_2+\mathcal{O}(1)$, *where* $\sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$;

(v) as $\zeta\to\infty$, $\zeta\in\mathbb{C}\setminus\sigma_c$, $m(x,t;\zeta)=\mathrm{I}+\mathcal{O}(\zeta^{-1})$, *where* $\mathrm{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$;

(vi) $m(x,t;\zeta)=\sigma_1\overline{m(x,t;\overline{\zeta})}\,\sigma_1$ *and* $m(x,t;\frac{1}{\zeta})=\zeta m(x,t;\zeta)\sigma_2$, *where* $\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Defining

$$u(x,t) := \mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_c}}(\zeta(m(x,t;\zeta)-\mathrm{I}))_{12} \tag{3}$$

$$\int_{+\infty}^{x}(|u(\xi,t)|^2-1)\,\mathrm{d}\xi := -\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_c}}(\zeta(m(x,t;\zeta)-\mathrm{I}))_{11} \tag{4}$$

$u(x,t)$ solves the Cauchy problem for the D_fNLSE, and, $\forall\, t\in\mathbb{R}$, $u(x,t)-u(\pm\infty,t)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R})$.

The solution of the RHP for $m(x,t;\zeta)\colon\mathbb{C}\setminus\sigma_c\to\mathrm{M}_2(\mathbb{C})$ formulated in lemma 1 is given by the following ordered product:

$$m(x,t;\zeta)=(\mathrm{I}+\Delta_o(x,t)\zeta^{-1})m^c(x,t;\zeta) \tag{5}$$

where $\Delta_o(x,t)$ ($\in GL(2,\mathbb{C})$) is obtained from the determining relation $\Delta_o(x,t)m^c(x,t;0)=\sigma_2$, $\Delta_o(x,t)=\sigma_1\overline{\Delta_o(x,t)}\,\sigma_1$ and $m^c(x,t;\zeta)\colon\mathbb{C}\setminus\sigma_c\to SL(2,\mathbb{C})$ solves the following (normalized at ∞) RHP: (1) $m^c(x,t;\zeta)$ is piecewise holomorphic $\forall\,\zeta\in\mathbb{C}\setminus\sigma_c$; (2) $m^c_+(x,t;\zeta)=m^c_-(x,t;\zeta)\mathcal{G}(x,t;\zeta)$, $\zeta\in\mathbb{R}$, where $\mathcal{G}(x,t;\zeta)$ is defined in lemma 1(ii); (3) $m^c(x,t;\zeta) =_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_c}} \mathrm{I}+\mathcal{O}(\zeta^{-1})$ and (4) $m^c(x,t;\zeta)$ satisfies the symmetry reduction $m^c(x,t;\zeta)=\sigma_1\overline{m^c(x,t;\overline{\zeta})}\,\sigma_1$ and the condition $(m^c(x,t;0)\sigma_2)^2=\mathrm{I}$.

The solution framework for RHPs of the type stated above for $m^c(x,t;\zeta)$ is the Beals–Coifman (BC) construction [21], a succinct synopsis of which follows (explicit x,t dependences are temporarily suppressed). Let $\Gamma^\sharp$, as a closed set, be the union of finitely many oriented simple piecewise-smooth arcs. Denote the set of all self-intersections of $\Gamma^\sharp$ by $\hat{\Gamma}^\sharp$ (with $\mathrm{card}(\hat{\Gamma}^\sharp)<\infty$). Set $\tilde{\Gamma}^\sharp:=\Gamma^\sharp\setminus\hat{\Gamma}^\sharp$. The BC formulation for the solution of a matrix RHP on an oriented contour $\Gamma^\sharp$ consists of finding an $\mathrm{M}_2(\mathbb{C})$-valued function $\mathcal{X}(\lambda)$ such that (1) $\mathcal{X}(\lambda)$ is piecewise holomorphic $\forall\,\lambda\in\mathbb{C}\setminus\Gamma^\sharp$, (2) $\mathcal{X}_+(\lambda)=\mathcal{X}_-(\lambda)\upsilon(\lambda)$, $\lambda\in\tilde{\Gamma}^\sharp$, for some 'jump' matrix $\upsilon(\lambda)\colon\tilde{\Gamma}^\sharp\to GL(2,\mathbb{C})$, and (3) uniformly as $\lambda\to\infty$, $\lambda\in\mathbb{C}\setminus\Gamma^\sharp$, $\mathcal{X}(\lambda)=\mathrm{I}+\mathcal{O}(\lambda^{-1})$. Let $\upsilon(\lambda):=(\mathrm{I}-w_-(\lambda))^{-1}(\mathrm{I}+w_+(\lambda))$, $\lambda\in\tilde{\Gamma}^\sharp$, be a factorization for $\upsilon(\lambda)$, where $w_\pm(\lambda)$ are some upper/lower, or lower/upper, triangular nilpotent matrices with degree of nilpotency 2, and $w_\pm(\lambda)\in\bigcap_{p\in\{2,\infty\}}\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\tilde{\Gamma}^\sharp)$ [2] (if $\tilde{\Gamma}^\sharp$ is unbounded, one requires that $w_\pm(\lambda) =_{\substack{\lambda\to\infty\\ \lambda\in\tilde{\Gamma}^\sharp}} \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$). Define $w(\lambda):=w_+(\lambda)+w_-(\lambda)$, and introduce the Cauchy operators on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^\sharp)$, $(C_\pm f)(\lambda):=\lim_{\substack{\lambda'\to\lambda\\ \lambda'\in\pm\,\text{side of}\,\Gamma^\sharp}}\int_{\Gamma^\sharp}\frac{f(z)}{(z-\lambda')}\frac{\mathrm{d}z}{2\pi\mathrm{i}}$, where $\lambda'\to\lambda$, $\lambda'\in\pm$ side of $\Gamma^\sharp$, denotes the non-tangential limits from the $\pm$ sides of $\Gamma^\sharp$ at $\lambda\in\Gamma^\sharp$. Introduce the BC operator on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)$, $C_w f:=C_+(fw_-)+C_-(fw_+)$. Re-introducing x,t dependences, specializing the BC formulation to the solution of the RHP for $m^c(x,t;\zeta)$ and defining $\mathcal{G}(x,t;\zeta):=(\mathrm{I}-w^{\mathcal{G}}_-(x,t;\zeta))^{-1}(\mathrm{I}+w^{\mathcal{G}}_+(x,t;\zeta))$, $\zeta\in\sigma_c$, the integral representation for $m^c(x,t;\zeta)$ is given by the following lemma 2.

[2] $\|\mathcal{F}(\cdot)\|_{\cap_{p\in\{2,\infty\}}\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(*)}:=\sum_{p\in\{2,\infty\}}\|\mathcal{F}(\cdot)\|_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(*)}$.

Lemma 2 [21]. *Let*

$$\mu^{\mathcal{G}}(x,t;\zeta) = m_{+}^{c}(x,t;\zeta)(\mathrm{I}+w_{+}^{\mathcal{G}}(x,t;\zeta))^{-1} = m_{-}^{c}(x,t;\zeta)(\mathrm{I}-w_{-}^{\mathcal{G}}(x,t;\zeta))^{-1}.$$

If $\mu^{\mathcal{G}}(x,t;\zeta) \in \mathrm{I}+\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\sigma_c) := \{\mathrm{I}+h;\ h \in \mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\sigma_c)\}$ *solves the linear singular integral equation*

$$(\mathbf{1}-C_{w^{\mathcal{G}}})(\mu^{\mathcal{G}}(x,t;\zeta)-\mathrm{I}) = C_{w^{\mathcal{G}}}\mathrm{I} = C_{+}(w_{-}^{\mathcal{G}}(x,t;\zeta)) + C_{-}(w_{+}^{\mathcal{G}}(x,t;\zeta)) \qquad \zeta \in \sigma_c,$$

where $\mathbf{1}$ *is the identity operator on* $\mathrm{I}+\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\sigma_c)$*, then the solution of the RHP for* $m^{c}(x,t;\zeta)$ *is*

$$m^{c}(x,t;\zeta) = \mathrm{I} + \int_{\sigma_c} \frac{\mu^{\mathcal{G}}(x,t;z)w^{\mathcal{G}}(x,t;z)}{(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}} \qquad \zeta \in \mathbb{C}\setminus\sigma_c$$

where $\mu^{\mathcal{G}}(x,t;\zeta) := ((\mathbf{1}-C_{w^{\mathcal{G}}})^{-1}\mathrm{I})(x,t;\zeta)$*, and* $w^{\mathcal{G}}(x,t;\zeta) := \sum_{l\in\{\pm\}} w_{l}^{\mathcal{G}}(x,t;\zeta)$.

Several key solvability/existence results for operators of the type $(\mathbf{1}-C_{\star})^{-1}$ related to integrable NLEEs can be found in the works of Zhou [22]. From lemma 2, the ordered factorization of equation (5) and equation (3), one shows that

$$u(x,t) = \mathrm{i}(\Delta_o(x,t))_{12} + \int_{\sigma_c} (\mu^{\mathcal{G}}(x,t;z))_{11}\,\overline{r(\overline{z})}\,\exp\left(-2\mathrm{i}t\theta^{u}(z)\right)\frac{\mathrm{d}z}{2\pi}$$
$$\theta^{u}(\zeta) := \frac{1}{2}\left(\zeta-\frac{1}{\zeta}\right)\left(z_o+\zeta+\frac{1}{\zeta}\right) \qquad z_o := \frac{x}{t} \tag{6}$$

(with an analogous expression for $\int_{+\infty}^{x}(|u(\xi,t)|^2-1)\,\mathrm{d}\xi$), where $(\star)_{ij}$, $i,j \in \{1,2\}$, denotes the $(i\,j)$-element of $\star$. *A priori*, no explicit information regarding the resolvent kernel, $\mu^{\mathcal{G}}(x,t;\zeta)$, is available: the remedy to this is achieved via the application of the Deift–Zhou (DZ) nonlinear steepest-descent procedure [23]. The DZ procedure begins by examining $\{z;\ \partial_{\zeta}\theta^{u}(\zeta)|_{\zeta=z}=0\}$, namely, the saddle/stationary phase point(s) of the phase function, $\theta^{u}(\zeta)$. The following cases evince themselves: (i) $t \to \pm\infty$ and $x \to \mp\infty$ such that $z_o < -2$, $\partial_{\zeta}\theta^{u}(\zeta) = \zeta^{-3}(\zeta-\zeta_1)(\zeta-\zeta_2)(\zeta-\zeta_3)(\zeta-\zeta_4)$, where $\{\zeta_i\}_{i=1}^{4}$ are given in theorem 1, equations (15) and (16) of [20]; (ii) $t \to \pm\infty$ and $x \to \pm\infty$ such that $z_o > 2$, $\partial_{\zeta}\theta^{u}(\zeta) = \zeta^{-3}(\zeta-\aleph_1)(\zeta-\aleph_2)(\zeta-\aleph_3)(\zeta-\aleph_4)$, where $\{\aleph_i\}_{i=1}^{4}$ are given in theorem 1, equation (23) of [20]; (iii) $t \to \pm\infty$ and $x \to \mp\infty$ (respectively $x \to \pm\infty$) such that $z_o \in (-2,0)$ (respectively $z_o \in (0,2)$), $\partial_{\zeta}\theta^{u}(\zeta) = \zeta^{-3}(\zeta-\zeta_1^{\sharp})(\zeta-\overline{\zeta_1^{\sharp}})(\zeta-\zeta_3^{\sharp})(\zeta-\overline{\zeta_3^{\sharp}})$, where $\zeta_1^{\sharp} := \frac{1}{2}(-a_1+\mathrm{i}(4-a_1^2)^{1/2}) = \exp(\mathrm{i}\hat{\varphi}_1)$, $\hat{\varphi}_1 = \arctan\left(\frac{(4-a_1^2)^{1/2}}{|a_1|}\right) \in (0,\frac{\pi}{2})$, with $a_1 = \frac{1}{4}(z_o-(z_o^2+32)^{1/2})$, $a_1 < 0$ and $|a_1| < 2$, and $\zeta_3^{\sharp} := \frac{1}{2}(-a_2+\mathrm{i}(4-a_2^2)^{1/2}) = \exp(\mathrm{i}\hat{\varphi}_3)$, $\hat{\varphi}_3 = -\arctan\left(\frac{(4-a_2^2)^{1/2}}{|a_2|}\right) \in (\frac{\pi}{2},\pi)$, with $a_2 = \frac{1}{4}(z_o+(z_o^2+32)^{1/2})$, $a_2 > 0$ and $|a_2| < 2$; (iv) $t \to \pm\infty$ and $x \to \mp\infty$ (respectively $x \to \pm\infty$) such that $z_o \to 0^{-}$ (respectively $z_o \to 0^{+}$), $\partial_{\zeta}\theta^{u}(\zeta) = \zeta^{-3}(\zeta-\exp(\mathrm{i}\pi/4))(\zeta-\exp(-\mathrm{i}\pi/4))(\zeta-\exp(3\pi\mathrm{i}/4))(\zeta-\exp(-3\pi\mathrm{i}/4))$; (v) $t \to \pm\infty$ and $x \to \mp\infty$ such that $z_o = -2$, $\partial_{\zeta}\theta^{u}(\zeta) = \zeta^{-3}(\zeta-1)^2(\zeta-\exp(2\pi\mathrm{i}/3))(\zeta-\exp(-2\pi\mathrm{i}/3))$, and (vi) $t \to \pm\infty$ and $x \to \pm\infty$ such that $z_o = 2$, $\partial_{\zeta}\theta^{u}(\zeta) = \zeta^{-3}(\zeta+1)^2(\zeta-\exp(\mathrm{i}\pi/3))(\zeta-\exp(-\mathrm{i}\pi/3))$. Cases (i) and (ii) correspond to oscillatory asymptotics [20], cases (iii) and (iv) give rise to exponentially decaying asymptotics, and cases (v) and (vi) give rise to asymptotics which are related to those of the transcendent of the Painlevé II equation (PII) [24–26]. In this Letter, results for case (iv) are presented (see theorem 1).

Hereafter, and without loss of generality, the cases $t \to \pm\infty$ and $x \to \mp\infty$ such that $z_o \to 0^{-}$ are discussed (the two remaining cases are analogous). Note that the 'symbol' $\underline{c}$, appearing in the various error estimates, denotes a bounded $\mathbb{C}\setminus\{0\}$-valued constant. One proves, with the help of the second resolvent identity, that, uniformly for $\zeta \in \sigma_c$, as $t \to \pm\infty$ and $x \to \mp\infty$

such that $z_o \to 0^-$, $\mu^{\mathcal{G}}(x,t;\zeta) := ((\mathbf{1} - C_{w^{\mathcal{G}}})^{-1}\mathrm{I})(x,t;\zeta) = \mathrm{I} + ((\mathbf{1} - C_{w^{\mathcal{G}}})^{-1}C_{w^{\mathcal{G}}}\mathrm{I})(x,t;\zeta)$ has the following estimates:

$$\mu^{\mathcal{G}}(x,t;\zeta) = \mathrm{I} + \mathcal{O}\left(\frac{\underline{c}\,\exp(-2|t|)}{\sqrt{|t|}}\right). \tag{7}$$

The implementation of the DZ procedure, complemented by the estimates of equation (7), the asymptotic solution of the system of linear singular integral equations of lemma 2 and the determining relation $\Delta_o(x,t)m^c(x,t;0) = \sigma_2$, leads to the following results.

Lemma 3. *Let ε be an arbitrarily fixed, sufficiently small positive real number, and, for $\lambda \in \mathfrak{J} := \{(s_1)^{\pm 1}, (s_2)^{\pm 1}\}$, where $s_1 := \exp(\mathrm{i}\pi/4)$ and $s_2 := \exp(3\pi\mathrm{i}/4)$, set $\mathbb{U}(\lambda;\varepsilon) := \{z;\ |z-\lambda| < \varepsilon\}$. Then, for $r(s_1) = \exp(\mathrm{i}\pi/2)|r(s_1)|$, $r(\overline{s_2}) = \exp(-\mathrm{i}\pi/2)|r(\overline{s_2})|$, $0 < r(s_2)\overline{r(\overline{s_2})} < 1$ and $\zeta \in \mathbb{C} \setminus \bigcup_{\lambda \in \mathfrak{J}} \mathbb{U}(\lambda;\varepsilon)$, as $t \to +\infty$ and $x \to -\infty$ such that $z_o \to 0^-$, $m^c(\zeta) := m^c(x,t;\zeta)$ has the following asymptotics:*

$$m^c_{11}(\zeta) = \delta(\zeta)\left(1 + \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - s_1)} + \frac{\underline{c}}{(\zeta - \overline{s_2})}\right)\frac{\exp(-4t)}{t}\right)\right)$$

$$\begin{aligned} m^c_{12}(\zeta) = -\frac{1}{\delta(\zeta)}\Bigg(&\frac{\exp\left(-2t - \sqrt{2}\int_{-\infty}^0 \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(-\mathrm{i}\left(\frac{\pi}{4} + \sqrt{2}\int_{-\infty}^0 \frac{(\sqrt{2}\mu-1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{t}\,(|r(s_1)|)^{-1}(\zeta - \overline{s_1})} \\ &+ \frac{\exp\left(-2t + \sqrt{2}\int_{-\infty}^0 \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(\mathrm{i}\left(\frac{3\pi}{4} - \sqrt{2}\int_{-\infty}^0 \frac{(\sqrt{2}\mu+1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{t}\,(|r(\overline{s_2})|)^{-1}(1 - r(s_2)\overline{r(\overline{s_2})})(\zeta - s_2)} \\ &+ \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - \overline{s_1})} + \frac{\underline{c}}{(\zeta - s_2)}\right)\frac{\exp(-4t)}{t}\right)\Bigg) \end{aligned}$$

$$\begin{aligned} m^c_{21}(\zeta) = -\delta(\zeta)\Bigg(&\frac{\exp\left(-2t - \sqrt{2}\int_{-\infty}^0 \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(\mathrm{i}\left(\frac{\pi}{4} + \sqrt{2}\int_{-\infty}^0 \frac{(\sqrt{2}\mu-1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{t}\,(|r(s_1)|)^{-1}(\zeta - s_1)} \\ &+ \frac{\exp\left(-2t + \sqrt{2}\int_{-\infty}^0 \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(\mathrm{i}\left(-\frac{3\pi}{4} + \sqrt{2}\int_{-\infty}^0 \frac{(\sqrt{2}\mu+1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{t}\,(|r(\overline{s_2})|)^{-1}(1 - r(s_2)\overline{r(\overline{s_2})})(\zeta - \overline{s_2})} \\ &+ \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - s_1)} + \frac{\underline{c}}{(\zeta - \overline{s_2})}\right)\frac{\exp(-4t)}{t}\right)\Bigg), \end{aligned}$$

$$m^c_{22}(\zeta) = \frac{1}{\delta(\zeta)}\left(1 + \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - \overline{s_1})} + \frac{\underline{c}}{(\zeta - s_2)}\right)\frac{\exp(-4t)}{t}\right)\right)$$

where $\delta(\zeta) := \exp\left(\int_{-\infty}^0 \frac{\ln(1-|r(\mu)|^2)}{(\mu-\zeta)}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\right)$, with $\delta(\zeta)\overline{\delta(\overline{\zeta})} = 1$ and $\delta(\zeta)\delta(\frac{1}{\zeta}) = \delta(0)$, $\exists\, M^+ \in \mathbb{R}_{>0}$ (and bounded) such that $\sup_{\zeta \in \mathbb{C}\setminus\cup_{\lambda\in\mathfrak{J}}\mathbb{U}(\lambda;\varepsilon)} |(\zeta - \hat{\underline{\xi}})^{-1}| \leqslant M^+$, $\hat{\underline{\xi}} \in \mathfrak{J}$, $m^c(\zeta) = \sigma_1\overline{m^c(\overline{\zeta})}\,\sigma_1$ and $(m^c(0)\sigma_2)^2 = \mathrm{I}$.

Proposition 2. *As $t \to +\infty$ and $x \to -\infty$ such that $z_o \to 0^-$,*

$$(\Delta_o(x,t))_{11} = \frac{\mathrm{i}\exp(-2t - \tilde{\mathfrak{c}}_+)\mathfrak{b}_+}{2\sqrt{t}}\cosh\left(\tilde{\mathfrak{c}}_- - \ln\mathfrak{b}_-\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4t)}{t}\right)$$

$$(\Delta_o(x,t))_{12} = \exp\left(-\mathrm{i}\left(\frac{\pi}{2} + \int_{-\infty}^0 \frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}\right)\right)\left(1 + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4t)}{t}\right)\right)$$

$$(\Delta_o(x,t))_{21} = \exp\left(\mathrm{i}\left(\frac{\pi}{2} + \int_{-\infty}^0 \frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}\right)\right)\left(1 + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4t)}{t}\right)\right)$$

$$(\Delta_o(x,t))_{22} = -\frac{\mathrm{i}\exp(-2t - \tilde{\mathfrak{c}}_+)\mathfrak{b}_+}{2\sqrt{t}}\cosh\left(\tilde{\mathfrak{c}}_- - \ln\mathfrak{b}_-\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4t)}{t}\right)$$

where $\tilde{\mathfrak{c}}_\pm$ and $\mathfrak{b}_\pm$ are defined in theorem 1, equations (12) and (13).

Lemma 4. *Let ε be an arbitrarily fixed, sufficiently small positive real number, and, for $\lambda \in \mathfrak{J} := \{(s_1)^{\pm 1}, (s_2)^{\pm 1}\}$, where $s_1 := \exp(\mathrm{i}\pi/4)$ and $s_2 := \exp(3\pi\mathrm{i}/4)$, set $\mathbb{U}(\lambda; \varepsilon) := \{z;\ |z - \lambda| < \varepsilon\}$. Then, for $r(\overline{s_1}) = \exp(\mathrm{i}\pi/2)|r(\overline{s_1})|$, $r(s_2) = \exp(-\mathrm{i}\pi/2)|r(s_2)|$, $0 < r(s_1)\overline{r(\overline{s_1})} < 1$ and $\zeta \in \mathbb{C} \setminus \bigcup_{\lambda \in \mathfrak{J}} \mathbb{U}(\lambda; \varepsilon)$, as $t \to -\infty$ and $x \to +\infty$ such that $z_o \to 0^-$, $m^c(\zeta) := m^c(x, t; \zeta)$ has the following asymptotics:*

$$m^c_{11}(\zeta) = \tilde{\delta}(\zeta)\left(1 + \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - \overline{s_1})} + \frac{\underline{c}}{(\zeta - s_2)}\right)\frac{\exp(-4|t|)}{t}\right)\right)$$

$$\begin{aligned} m^c_{12}(\zeta) = -\frac{1}{\tilde{\delta}(\zeta)}\Bigg(& \frac{\exp\left(-2|t| + \sqrt{2}\int_0^{+\infty} \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(\mathrm{i}\left(\frac{\pi}{4} - \sqrt{2}\int_0^{+\infty} \frac{(\sqrt{2}\mu-1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{|t|}\,(|r(\overline{s_1})|)^{-1}(1 - r(s_1)\overline{r(\overline{s_1})})(\zeta - s_1)} \\ & + \frac{\exp\left(-2|t| - \sqrt{2}\int_0^{+\infty} \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(-\mathrm{i}\left(\frac{3\pi}{4} + \sqrt{2}\int_0^{+\infty} \frac{(\sqrt{2}\mu+1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{|t|}\,(|r(s_2)|)^{-1}(\zeta - \overline{s_2})} \\ & + \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - s_1)} + \frac{\underline{c}}{(\zeta - \overline{s_2})}\right)\frac{\exp(-4|t|)}{t}\right)\Bigg) \end{aligned}$$

$$\begin{aligned} m^c_{21}(\zeta) = -\tilde{\delta}(\zeta)\Bigg(& \frac{\exp\left(-2|t| + \sqrt{2}\int_0^{+\infty} \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(-\mathrm{i}\left(\frac{\pi}{4} - \sqrt{2}\int_0^{+\infty} \frac{(\sqrt{2}\mu-1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{|t|}\,(|r(\overline{s_1})|)^{-1}(1 - r(s_1)\overline{r(\overline{s_1})})(\zeta - \overline{s_1})} \\ & + \frac{\exp\left(-2|t| - \sqrt{2}\int_0^{+\infty} \frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\exp\left(\mathrm{i}\left(\frac{3\pi}{4} + \sqrt{2}\int_0^{+\infty} \frac{(\sqrt{2}\mu+1)\ln(1-|r(\mu)|^2)}{(\sqrt{2}\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi}\right)\right)}{4\sqrt{|t|}\,(|r(s_2)|)^{-1}(\zeta - s_2)} \\ & + \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - \overline{s_1})} + \frac{\underline{c}}{(\zeta - s_2)}\right)\frac{\exp(-4|t|)}{t}\right)\Bigg) \end{aligned}$$

$$m^c_{22}(\zeta) = \frac{1}{\tilde{\delta}(\zeta)}\left(1 + \mathcal{O}\left(\left(\frac{\underline{c}}{(\zeta - s_1)} + \frac{\underline{c}}{(\zeta - \overline{s_2})}\right)\frac{\exp(-4|t|)}{t}\right)\right)$$

where $\tilde{\delta}(\zeta) := \exp\left(\int_0^{+\infty} \frac{\ln(1-|r(\mu)|^2)}{(\mu-\zeta)}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\right)$, *with* $\tilde{\delta}(\zeta)\overline{\tilde{\delta}(\overline{\zeta})} = 1$ *and* $\tilde{\delta}(\zeta)\tilde{\delta}(\frac{1}{\zeta}) = \tilde{\delta}(0)$, $\exists\ M^- \in \mathbb{R}_{>0}$ *(and bounded) such that* $\sup_{\zeta \in \mathbb{C} \setminus \cup_{\lambda \in \mathfrak{J}} \mathbb{U}(\lambda;\varepsilon)} |(\zeta - \hat{\underline{\zeta}})^{-1}| \leqslant M^-$, $\hat{\underline{\zeta}} \in \mathfrak{J}$, $m^c(\zeta) = \sigma_1\overline{m^c(\overline{\zeta})}\sigma_1$ *and* $(m^c(0)\sigma_2)^2 = \mathrm{I}$.

Proposition 3. *As $t \to -\infty$ and $x \to +\infty$ such that $z_o \to 0^-$,*

$$(\Delta_o(x,t))_{11} = \frac{\mathrm{i}\exp(-2|t| + \hat{\mathfrak{c}}_+)\mathfrak{d}_+}{2\sqrt{|t|}}\cosh\left(\hat{\mathfrak{c}}_- - \ln\mathfrak{d}_-\right) + \mathcal{O}\left(\frac{\underline{c}\exp(-4|t|)}{t}\right)$$

$$(\Delta_o(x,t))_{12} = \exp\left(-\mathrm{i}\left(\frac{\pi}{2} + \int_0^{+\infty}\frac{\ln(1 - |r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}\right)\right)\left(1 + \mathcal{O}\left(\frac{\underline{c}\exp(-4|t|)}{t}\right)\right)$$

$$(\Delta_o(x,t))_{21} = \exp\left(\mathrm{i}\left(\frac{\pi}{2} + \int_0^{+\infty}\frac{\ln(1 - |r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}\right)\right)\left(1 + \mathcal{O}\left(\frac{\underline{c}\exp(-4|t|)}{t}\right)\right)$$

$$(\Delta_o(x,t))_{22} = -\frac{\mathrm{i}\exp(-2|t| + \hat{\mathfrak{c}}_+)\mathfrak{d}_+}{2\sqrt{|t|}}\cosh\left(\hat{\mathfrak{c}}_- - \ln\mathfrak{d}_-\right) + \mathcal{O}\left(\frac{\underline{c}\exp(-4|t|)}{t}\right)$$

where $\hat{\mathfrak{c}}_\pm$ and $\mathfrak{d}_\pm$ are defined in theorem 1, equations (17) and (18).

From lemmae 3 and 4, propositions 2 and 3, equations (3) and (4), the trace identity [20] $\int_{-\infty}^{+\infty}(|u(\xi,t)|^2 - 1)\,\mathrm{d}\xi = -\int_{-\infty}^{+\infty}\ln(1 - |r(\mu)|^2)\,\frac{\mathrm{d}\mu}{2\pi}$ and an analogous treatment for the two remaining cases, one arrives at the following.

Theorem 1. *For $r(\zeta) \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}) \cap \{h(z);\ \|h(\cdot)\|_{\mathcal{L}^\infty(\mathbb{R})} := \sup_{z\in\mathbb{R}}|h(z)| < 1\}$, and having an analytic continuation to $\mathbb{C}\setminus\mathbb{R}$, let $m(x, t; \zeta)$ be the solution of the RHP formulated in lemma 1. Let $u(x, t)$, the solution of the Cauchy problem for the D_fNLSE with finite-density initial data*

$u(x,0) := u_o(x) =_{x\to\pm\infty} u_o(\pm\infty)(1+\mathrm{o}(1))$, where $u_o(\pm\infty) := \exp\left(\frac{\mathrm{i}(1\mp 1)\theta}{2}\right)$, $0 \leqslant \theta = -\int_{-\infty}^{+\infty} \frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi} < 2\pi$, $u_o(x) \in \boldsymbol{C}^{\infty}(\mathbb{R})$ and $u_o(x) - u_o(\pm\infty) \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$, be defined by equation (3), and $\int_{+\infty}^{x}(|u(\xi,t)|^2-1)\,\mathrm{d}\xi$ be defined by equation (4). Set $s_1 := \exp(\mathrm{i}\pi/4)$ and $s_2 := \exp(3\pi\mathrm{i}/4)$. Then (i) for $r(s_1) = \exp(\mathrm{i}\pi/2)|r(s_1)|$, $r(\overline{s_2}) = \exp(-\mathrm{i}\pi/2)|r(\overline{s_2})|$ and $0 < r(s_2)\overline{r(\overline{s_2})} < 1$, as $t \to +\infty$ and $x \to -\infty$ such that $z_o := x/t \to 0^-$,

$$u(x,t)=\exp(-\mathrm{i}\psi^+(1))\left(1+\frac{\exp(\mathrm{i}\pi/4)\exp(-2t-\tilde{\mathfrak{c}}_+)\mathfrak{b}_+}{2\sqrt{t}}\sinh\left(\tilde{\mathfrak{c}}_- - \ln\mathfrak{b}_-\right)+\mathcal{O}\left(\frac{\underline{c}\exp(-4t)}{t}\right)\right) \tag{8}$$

$$\int_{+\infty}^{x}(|u(\xi,t)|^2-1)\,\mathrm{d}\xi = \psi^+(0)+\frac{\exp(-2t-\tilde{\mathfrak{c}}_+)\mathfrak{b}_+}{2\sqrt{t}}\cosh\left(\tilde{\mathfrak{c}}_- - \ln\mathfrak{b}_-\right)+\mathcal{O}\left(\frac{\underline{c}\exp(-4t)}{t}\right) \tag{9}$$

$$\int_{-\infty}^{x}(|u(\xi,t)|^2-1)\,\mathrm{d}\xi = -\psi^-(0)+\frac{\exp(-2t-\tilde{\mathfrak{c}}_+)\mathfrak{b}_+}{2\sqrt{t}}\cosh\left(\tilde{\mathfrak{c}}_- - \ln\mathfrak{b}_-\right)+\mathcal{O}\left(\frac{\underline{c}\exp(-4t)}{t}\right) \tag{10}$$

where

$$\begin{aligned}\psi^+(l) &:= \int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{\mu^l}\frac{\mathrm{d}\mu}{2\pi}\\ \psi^-(l) &:= \int_{0}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu^l}\frac{\mathrm{d}\mu}{2\pi} \qquad l\in\{0,1\}\end{aligned} \tag{11}$$

$$\tilde{\mathfrak{c}}_\pm := \frac{1}{\sqrt{2}}\int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\,\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi} \mp \frac{1}{\sqrt{2}}\int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\,\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi} \tag{12}$$

$$\mathfrak{b}_+ := \left(\frac{|r(s_1)||r(\overline{s_2})|}{(1-r(s_2)\overline{r(\overline{s_2})})}\right)^{1/2} \qquad \mathfrak{b}_- := \left(\frac{|r(s_1)|(1-r(s_2)\overline{r(\overline{s_2})})}{|r(\overline{s_2})|}\right)^{1/2} \tag{13}$$

(ii) for $r(\overline{s_1}) = \exp(\mathrm{i}\pi/2)|r(\overline{s_1})|$, $r(s_2) = \exp(-\mathrm{i}\pi/2)|r(s_2)|$ and $0 < r(s_1)\overline{r(\overline{s_1})} < 1$, as $t \to -\infty$ and $x \to +\infty$ such that $z_o \to 0^-$,

$$\begin{aligned}u(x,t) = \exp(-\mathrm{i}\psi^-(1))&\left(1+\frac{\exp(-\mathrm{i}\pi/4)\exp(-2|t|+\hat{\mathfrak{c}}_+)\mathfrak{d}_+}{2\sqrt{|t|}}\sinh\left(\hat{\mathfrak{c}}_- - \ln\mathfrak{d}_-\right)\right.\\ &\left.+\mathcal{O}\left(\frac{\underline{c}\exp(-4|t|)}{t}\right)\right)\end{aligned} \tag{14}$$

$$\int_{+\infty}^{x}(|u(\xi,t)|^2-1)\mathrm{d}\xi = \psi^-(0)+\frac{\exp(-2|t|+\hat{\mathfrak{c}}_+)\mathfrak{d}_+}{2\sqrt{|t|}}\cosh\left(\hat{\mathfrak{c}}_- - \ln\mathfrak{d}_-\right)+\mathcal{O}\left(\frac{\underline{c}\exp(-4|t|)}{t}\right) \tag{15}$$

$$\int_{-\infty}^{x}(|u(\xi,t)|^2-1)\,\mathrm{d}\xi = -\psi^+(0)+\frac{\exp(-2|t|+\hat{\mathfrak{c}}_+)\mathfrak{d}_+}{2\sqrt{|t|}}\cosh\left(\hat{\mathfrak{c}}_- - \ln\mathfrak{d}_-\right)+\mathcal{O}\left(\frac{\underline{c}\exp(-4|t|)}{t}\right) \tag{16}$$

where

$$\hat{\mathfrak{c}}_\pm := \frac{1}{\sqrt{2}}\int_{0}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\,\mu-1)^2+1}\frac{\mathrm{d}\mu}{\pi} \mp \frac{1}{\sqrt{2}}\int_{0}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\sqrt{2}\,\mu+1)^2+1}\frac{\mathrm{d}\mu}{\pi} \tag{17}$$

$$\mathfrak{d}_+ := \left(\frac{|r(\overline{s_1})||r(s_2)|}{(1-r(s_1)\overline{r(\overline{s_1})})}\right)^{1/2} \qquad \mathfrak{d}_- := \left(\frac{|r(s_2)|(1-r(s_1)\overline{r(\overline{s_1})})}{|r(\overline{s_1})|}\right)^{1/2} \tag{18}$$

(iii) for $r(s_1) = \exp(-\mathrm{i}\pi/2)|r(s_1)|$, $r(\overline{s_2}) = \exp(\mathrm{i}\pi/2)|r(\overline{s_2})|$ and $0 < r(s_1)\overline{r(\overline{s_1})} < 1$, as $t \to +\infty$ and $x \to +\infty$ such that $z_o \to 0^+$,

$$u(x,t) = -\exp(-\mathrm{i}\psi^{+}(1))\left(1 + \frac{\exp(\mathrm{i}\pi/4)\exp(-2t-\tilde{\mathfrak{c}}_{+})\tilde{\mathfrak{g}}_{+}}{2\sqrt{t}}\sinh\left(\tilde{\mathfrak{c}}_{-} + \ln\tilde{\mathfrak{g}}_{-}\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4t)}{t}\right)\right) \tag{19}$$

$$\int_{+\infty}^{x}(|u(\xi,t)|^{2}-1)\,\mathrm{d}\xi = \psi^{-}(0) - \frac{\exp(-2t-\tilde{\mathfrak{c}}_{+})\tilde{\mathfrak{g}}_{+}}{2\sqrt{t}}\cosh\left(\tilde{\mathfrak{c}}_{-} + \ln\tilde{\mathfrak{g}}_{-}\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4t)}{t}\right) \tag{20}$$

$$\int_{-\infty}^{x}(|u(\xi,t)|^{2}-1)\,\mathrm{d}\xi = -\psi^{+}(0) - \frac{\exp(-2t-\tilde{\mathfrak{c}}_{+})\tilde{\mathfrak{g}}_{+}}{2\sqrt{t}}\cosh\left(\tilde{\mathfrak{c}}_{-} + \ln\tilde{\mathfrak{g}}_{-}\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4t)}{t}\right) \tag{21}$$

where

$$\tilde{\mathfrak{g}}_{+} := \left(\frac{|r(\overline{s_2})||r(s_1)|}{(1 - r(s_1)\overline{r(\overline{s_1})})}\right)^{1/2} \qquad \tilde{\mathfrak{g}}_{-} := \left(\frac{|r(\overline{s_2})|(1 - r(s_1)\overline{r(\overline{s_1})})}{|r(s_1)|}\right)^{1/2} \tag{22}$$

and (iv) for $r(\overline{s_1}) = \exp(-\mathrm{i}\pi/2)|r(\overline{s_1})|$, $r(s_2) = \exp(\mathrm{i}\pi/2)|r(s_2)|$ *and* $0 < r(s_2)\overline{r(\overline{s_2})} < 1$, *as* $t \to -\infty$ *and* $x \to -\infty$ *such that* $z_o \to 0^{+}$,

$$u(x,t) = -\exp(-\mathrm{i}\psi^{-}(1))\left(1 + \frac{\exp(-\mathrm{i}\pi/4)\exp(-2|t|+\hat{\mathfrak{c}}_{+})\hat{\mathfrak{g}}_{+}}{2\sqrt{|t|}}\sinh\left(\hat{\mathfrak{c}}_{-} + \ln\hat{\mathfrak{g}}_{-}\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4|t|)}{t}\right)\right) \tag{23}$$

$$\int_{+\infty}^{x}(|u(\xi,t)|^{2}-1)\,\mathrm{d}\xi = \psi^{+}(0) - \frac{\exp(-2|t|+\hat{\mathfrak{c}}_{+})\hat{\mathfrak{g}}_{+}}{2\sqrt{|t|}}\cosh\left(\hat{\mathfrak{c}}_{-} + \ln\hat{\mathfrak{g}}_{-}\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4|t|)}{t}\right) \tag{24}$$

$$\int_{-\infty}^{x}(|u(\xi,t)|^{2}-1)\,\mathrm{d}\xi = -\psi^{-}(0) - \frac{\exp(-2|t|+\hat{\mathfrak{c}}_{+})\hat{\mathfrak{g}}_{+}}{2\sqrt{|t|}}\cosh\left(\hat{\mathfrak{c}}_{-} + \ln\hat{\mathfrak{g}}_{-}\right) + \mathcal{O}\left(\frac{\underline{c}\,\exp(-4|t|)}{t}\right) \tag{25}$$

where

$$\hat{\mathfrak{g}}_{+} := \left(\frac{|r(s_2)||r(\overline{s_1})|}{(1 - r(s_2)\overline{r(\overline{s_2})})}\right)^{1/2} \qquad \hat{\mathfrak{g}}_{-} := \left(\frac{|r(\overline{s_1})|(1 - r(s_2)\overline{r(\overline{s_2})})}{|r(s_2)|}\right)^{1/2}. \tag{26}$$

The author is very grateful to the referee for a careful reading of the Letter and for many helpful suggestions.

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